12.501/12.201 Harmonic Analysis

Basic motivation:

De Bremaeker 88–95
Officer: 27–42.

1) Variation in scale – spectrum

e.g. gravity, heat flow, seismic velocity } distance

seismic response, attenuation, hearing } time

2) "Natural" specification:
Normal modes – string, sphere

3) Mathematical analysis – simplicity

4) "Filtering" – physical systems

Laplacian operator $\nabla^2 f$ – often used
($\nabla^2 f \sim \frac{\Delta \text{slope}}{\Delta x}$)

e.g.

gravity $\nabla^2 V = 0$ $\;= 4\pi G\rho$ Laplaces equ

viscous flow $\nabla^2 \psi = \omega$ Poisson's equ

$\nabla^2 \omega = \partial f/\partial x$

heat flow $\nabla^2 T = \frac{\partial T}{\partial t}$ Diffusion equ

wave equ $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$

(dimensionless)

Common method of solution: separation of variables

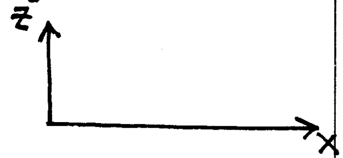
e.g. Cartesian $f = X(x)\, Y(y)\, Z(z)\, T(t)$

Cylindrical $f = R(r)\, \Theta(\theta)\, Z(z)\, T(t)$

Spherical $f = R(r)\, \Theta(\theta)\, \Phi(\varphi)\, T(t)$

Example: 2-D, Cartesian gravity

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} = 0$$



V - gravitational potential

set $V = X(x)\,Z(z)$

$$\frac{X''}{X} + \frac{Z''}{Z} = 0$$

try $\frac{X''}{X} = -k_n^2 \qquad \frac{Z''}{Z} = k_n^2 \rightarrow$ satisfies eqn

Solution: (assume periodic in x with $\lambda = L$)

$$V(x,z) = \sum_{n=0}^{\infty}\left[(a_n \cos k_n x + b_n \sin k_n x)\cdot (c_n e^{k_n z} + d_n e^{-k_n z})\right]$$

where $k_n = \frac{2\pi n}{L}$

Reasons: sines & cosines – orthogonal basis functions

any "well behaved" function can be expressed

In general: horizontal variations expressed in "harmonic functions"

Cartesian : sine, cosine
Cylindrical : Bessel functions
Sphere : Spherical Harmonics

Let $\underset{\sim}{x}$ be a position vector on a surface (e.g. (x, y), (r, θ), (θ, φ))

Let $\varphi_i(\underset{\sim}{x})$ be a harmonic function

Define a generalized Fourier Series:
For any "well behaved" function $f(\underset{\sim}{x})$, to arbitrary accuracy

$$f(\underset{\sim}{x}) = \sum_i a_i \varphi_i(\underset{\sim}{x})$$

Harmonic basis functions are:

Orthogonal

$$\int_{\text{unit cell}} \varphi_j(\underset{\sim}{x}) \varphi_k(\underset{\sim}{x})\, dA \propto \delta_{jk}$$

usually normalized

$$\int_{\text{unit cell}} \left(\varphi_i(\underset{\sim}{x})\right)^2 dA = c = \text{constant} \quad (1, \pi, 4\pi)$$

Recipe for a_i

$$\int_{\text{unit cell}} f(\underset{\sim}{x}) \varphi_k(\underset{\sim}{x})\, dA = \int_{\text{unit cell}} \sum_j a_j \varphi_j(\underset{\sim}{x}) \varphi_k(x)\, dA$$

$$= \sum_j a_j \int_{\text{unit cells}} \varphi_j(\underset{\sim}{x}) \varphi_k(\underset{\sim}{x})\, dA$$

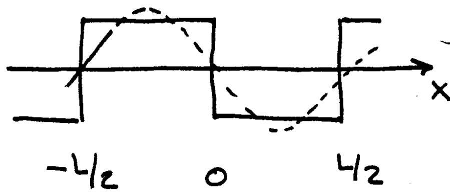
$$= c \sum_j a_j \delta_{jk}$$

$$= c\, a_k \Rightarrow a_k = \frac{1}{c} \int f(\underset{\sim}{x}) \varphi_k(\underset{\sim}{x})\, dA$$

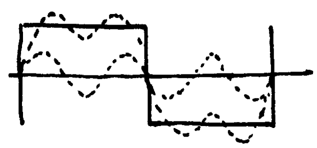
Example: square wave - analyze w/ sines & cosines

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2\pi}{L}nx + b_n \sin\frac{2\pi}{L}nx\right]$$ with, e.g.,

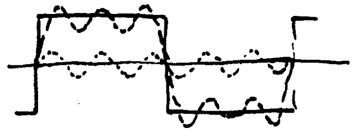
$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\frac{2\pi n x}{L}\,dx$$



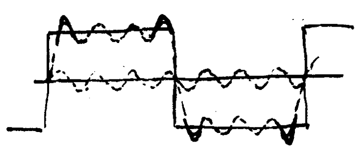
$$f(x) = -\frac{4}{\pi}\left(\sin\frac{2\pi x}{L}\right.$$



$$+\frac{1}{3}\sin\frac{6\pi x}{L}$$



$$+\frac{1}{5}\sin\frac{10\pi x}{L}$$



$$+\frac{1}{7}\sin\frac{14\pi x}{L}$$

$$+ \cdots\cdots$$

Note: odd function $\Rightarrow a_n = 0$

$$b_n = -\frac{4}{n\pi} \quad n \text{ odd}$$

$$= 0 \quad n \text{ even}$$

Useful concept:

rms coefficient $\quad s_n = \left(\frac{a_n^2 + b_n^2}{2}\right)^{1/2}$

power $\quad P_n = a_n^2 + b_n^2$

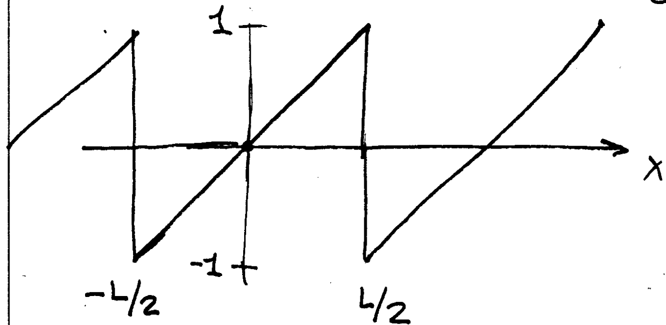
here $s_n \sim \frac{1}{n}$

Sawtooth wave –

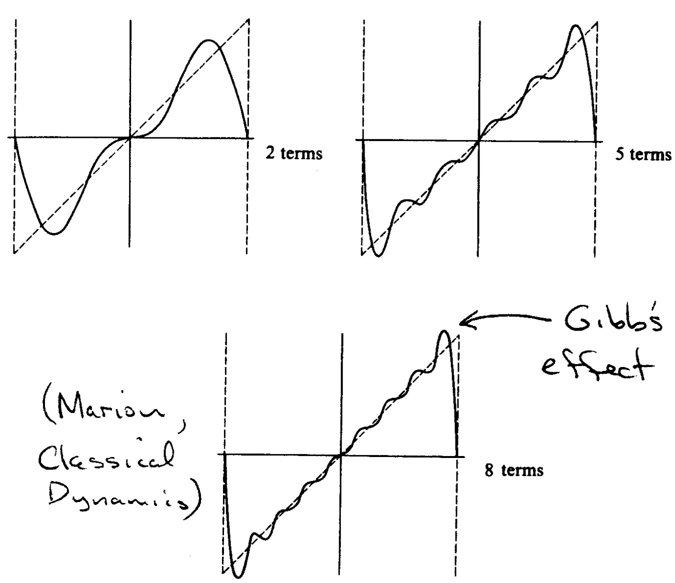
again, an odd function

$a_n = 0$

$b_n = \frac{2}{\pi n}$

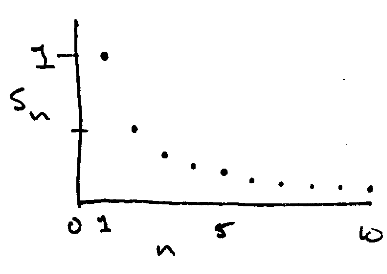


14.12 FOURIER ANALYSIS 469



(Marion, Classical Dynamics)

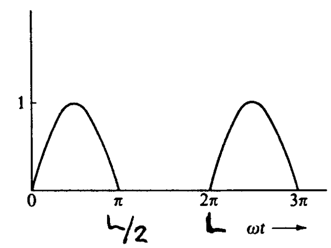
FIG. 14-9



again, $S_n \sim \frac{1}{n}$

A smoother function: "rectified" sine wave

(neither odd nor even $\Rightarrow$ both a_n & b_n)



The evaluation of the Fourier coefficients yields the following results:

$$
\left.
\begin{aligned}
a_0 &= \frac{2}{\pi} \\
a_1 &= 0 \\
a_r &= -\frac{1}{\pi}\left[\frac{\cos(r+1)\omega t}{2(r+1)} - \frac{\cos(r-1)\omega t}{2(r-1)}\right]_0^{\pi}, \qquad r \neq 0, 1 \\
b_1 &= \tfrac{1}{2} \\
b_r &= \frac{1}{2\pi}\left[\frac{\sin(r-1)\omega t}{r-1} - \frac{\sin(r+1)\omega t}{r+1}\right]_0^{\pi} = 0, \qquad r \neq 0, 1
\end{aligned}
\right\} \qquad (2)
$$

$$
\begin{cases} = 0 & r \text{ odd} \\ = \frac{1}{\pi}\left(\frac{1}{r+1} - \frac{1}{r-1}\right) & r \text{ even} \end{cases}
$$

The solid curve in Fig. 14-11 shows the function

$$f(t) = \tfrac{1}{2}a_0 + b_1 \sin \omega t + a_2 \cos 2\omega t + a_4 \cos 4\omega t \qquad (3)$$

$\sim \frac{1}{r^2}, \; r \gg 1$

and the dashed curve is the exact function. It is apparent that even the first few terms of the expansion give a fairly close approximation to the function. The original function in this case is "smoother" than the saw-toothed function of the preceding example, so the convergence of the Fourier series is much more rapid. This is a general result: a highly discontinuous function can be approximated with reasonable accuracy by a Fourier series only if a large number of terms is used.

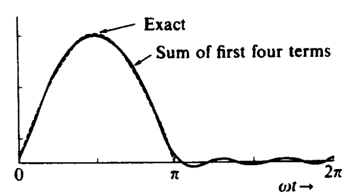
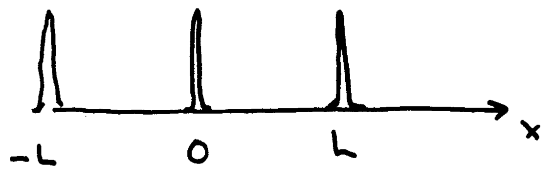


FIG. 14-11

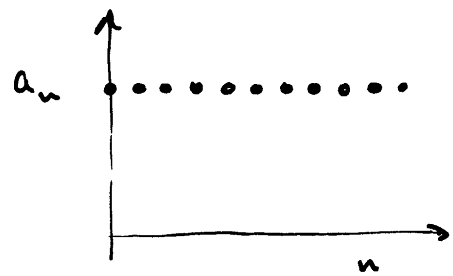
δ function ("spike" of unit area, width $\to 0$)

even $\Rightarrow b_n = 0$



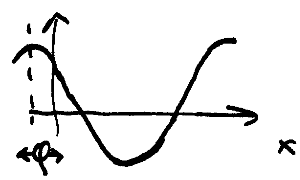
$a_n = \frac{2}{L}$

All coefficients are equal $\Rightarrow$ "white" spectrum



stopped here 1/9/86

Extensions:



1) Phase spectrum:

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \left(c_n \cos\left(\frac{2\pi}{L} n x + \phi_n\right)\right)$$

with $c_n = \sqrt{a_n^2 + b_n^2}$

$$\phi_n = \tan^{-1}\left(-b_n / a_n\right)$$

$\phi = -\pi/2 \Rightarrow a_n = 0$, just b_n

$\phi = \pi \Rightarrow a_n$ negative

2) Continuous functions ($L \to \infty$)

$$a(k) = \int_{-\infty}^{\infty} f(x) \cos(2\pi k x)\, dx$$

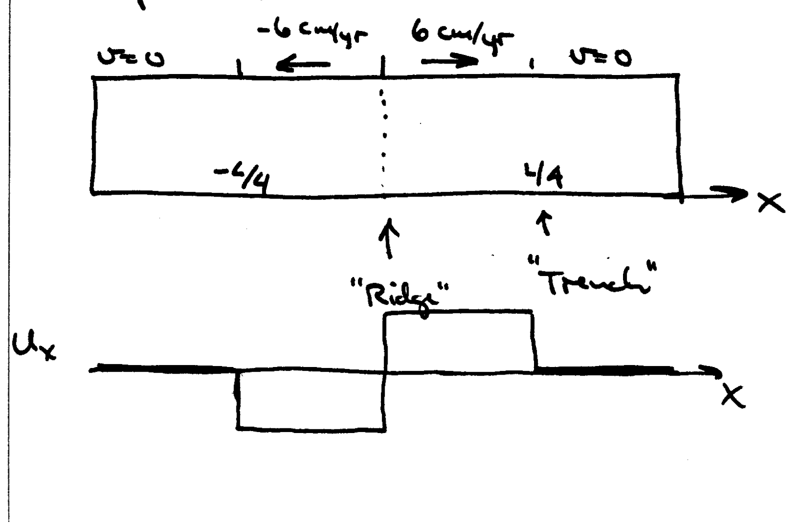
$$f(x) = 2 \int_{0}^{\infty} \left[a(k) \cos 2\pi k x + b(k) \sin 2\pi k x \right] dk$$

[note: other notations, more usual, more symmetric involve complex #'s]

In practice, fft used.

Example (with hazards!): Flow in mantle associated with plate motions
flow pattern, stress distribution, driving forces, ...

Conceptual model



Fourier analysis

velocity $v_x = \sum_n v_n \, v^n(z) \sin \frac{2\pi n x}{L}$ $\qquad v_n = \frac{2 v_p}{\pi n}\left(1-\cos\frac{\pi v}{2}\right.$

traction $\tau = \sum_n \tau_n \, \tau^n(z) \sin \frac{2\pi n x}{L}$ $\qquad \tau_{sn} \sim \frac{16\eta_0 V}{n\pi}\left(1-\cos\frac{n\pi}{2}\right.$

stream function $\psi = \sum_n \psi_n \, \psi^n(z) \sin \frac{2\pi n x}{L}$

$n_{max} = 30$

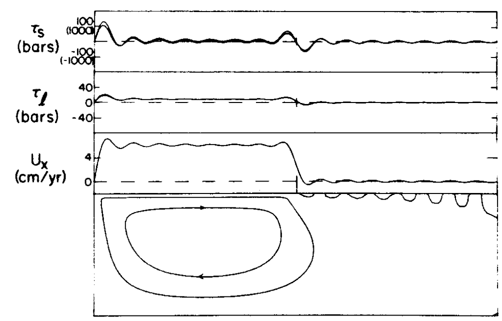


Fig. 2*a*. Surface tractions τ_s, tractions τ_l, at a depth of 64 km, surface velocities, and streamlines for a model in which a surface velocity of 6 cm/yr is imposed as a boundary condition on the oceanic plate on the left half of the box. No density contrasts are included. The viscosity structure consists of a 64-km-thick 10^{22} P layer overlying a 64-km-thick 4×10^{20} P low-viscosity layer overlying a 10^{22} P mantle. The stream function contour interval is 0.05.

Note: Gibbs effect ⇒ "unreal" (warts accentuated by "continuous" representation. Exact values at points at 1/30L spacing)

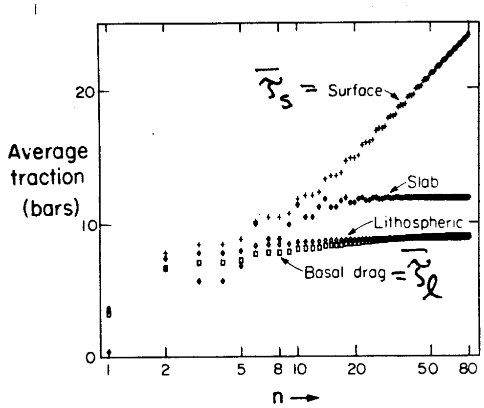


Fig. 3. Average magnitudes of shear tractions on the oceanic plate from Figure 2 as a function of the number of terms *n* retained in the Fourier series. Crosses represent the average tractions $\bar{\tau}_s$ at the surface for a velocity of 6 cm/yr, as in Figure 2*a*; boxes represent the tractions $\bar{\tau}_l$ for the model with no density contrasts in Figure 2*a*. Open diamonds represent $\bar{\tau}_s$ for lithospheric thickening as in Figure 2*b*, while solid diamonds represent the average tractions $\bar{\tau}_s$ from the subducted slab, as in Figure 2*c*.

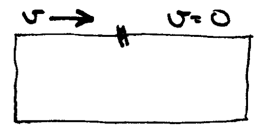
$\bar{\tau}_s$ blows up! $\sim \sum 1/n$

$\bar{\tau}_\ell$ does not

some "filtering" going on

"Filtering" — physics changes initial spectrum.

e.g. flow model

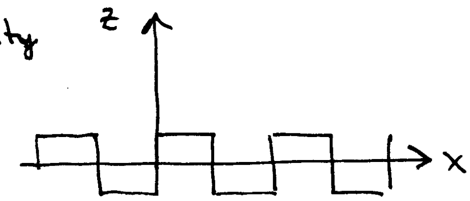


spectrum of velocity and surface tractions $\sim 1/n$

integral of surf traction $\sim \ln(n)$ (diverges)

integral of traction at some depth converges

gravity



$\Delta\rho_n(x, 0) \sim 1/n$

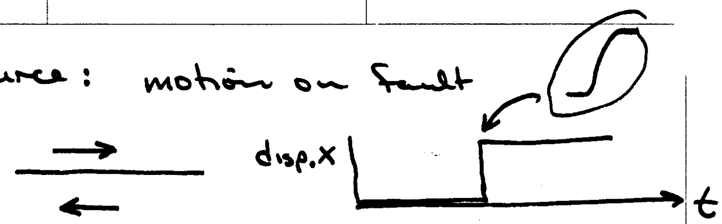
$\Delta g_n \sim \frac{1}{n} \exp(-k_n z)$ $\qquad k_n = \frac{2\pi n}{L}$

→ at high altitude, "smoother" field
Gibbs problem goes away

seismic source — high & low frequency
seismic attenuation
instrument response
→ "smooth" seismogram

building response, earth response
(Mexico city example)

Seismic source: motion on fault



$$x \cong \sum_n c_n \cos(\omega_n t + \varphi_n) \qquad S_x(\omega) \sim 1/\omega$$

$$v = \dot{x} \qquad S_v \sim \text{const}$$

$$a = \dot{v} \qquad S_a \sim \omega$$

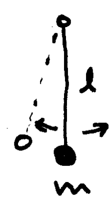
"Filters"

1) source

2) attenuation (Q) $\quad Q^{-1} \sim \Delta E / E$

3) structure

4) instrument



$$\omega \sim \sqrt{g/l}$$