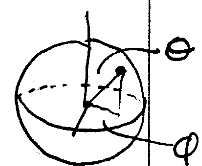
Spherical Harmonics

gravity field (e.g. $\Delta\rho$, Δg, ΔN
magnetic field
free oscillations
variations in seismic velocities
convection, ...



Introduce using gravity eqn

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2} = 0$$

Solve using separation of variables

$$V = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

choose $R(r) = r^{l}$ or $r^{-(l+1)}$

$$l(l+1) + \frac{1}{\sin\theta\,\Theta(\theta)}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\sin^2\theta\,\Phi(\varphi)}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

Note - only 3rd term has φ $\Rightarrow$ = const

$$\frac{d^2\Phi}{d\varphi^2} = k\,\Phi \Rightarrow \Phi = A\cos m\varphi + B\sin m\varphi$$

is a soln $(k = -m^2)$

then Θ eqn becomes Legendre's associated eqn

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

The solutions, for $m \le l$ is given in terms of Associated Legendre Polynomials

$$\Theta \equiv P_l^m(\cos\theta) = \frac{(1-\cos^2\theta)^{m/2}}{2^l\, l!} \frac{d^{l+m}(\cos^2\theta - 1)^l}{d(\cos\theta)^{l+m}}$$

& finally

$$V_l^m(r,\theta,\varphi) = \begin{bmatrix} r^l \\ r^{-(l+1)} \end{bmatrix} P_l^m(\cos\theta)\left[A_l^m \cos m\varphi + B_l^m \sin m\varphi\right]$$

$$\& \quad V = \sum_{l=0}^{\infty} \sum_{m=0}^{l} V_l^m$$

l = degree
m = order

For geophysical purposes, P_l^m's are best normalized to have rms = 1

In this form

$P_0^0 = 1$

$P_1^0 = \sqrt{3}\cos\theta$

$P_1^1 = \sqrt{3}\sin\theta$

$P_2^0 = \frac{\sqrt{5}}{2}(3\cos^2\theta - 1)$

$P_2^1 = \sqrt{\frac{5}{3}}\cos\theta\sin\theta$

$P_2^2 = \frac{\sqrt{15}}{2}\sin^2\theta$

$P_3^0 = \frac{\sqrt{7}}{2}(5\cos^3\theta - 3\cos\theta)$

$P_3^1 = \sqrt{\frac{21}{8}}(5\cos^2\theta - 1)\sin\theta$

$P_3^2 = \frac{\sqrt{105}}{2}\cos\theta\sin^2\theta$

$P_3^3 = \sqrt{\frac{35}{8}}\sin^3\theta$

The total function needed has both θ & ϕ dependence

$$Y_{lm}^{c} \equiv P_l^m(\cos\theta)\cos m\phi$$

$$Y_{lm}^{s} = P_l^m(\cos\theta)\sin m\phi$$

where $l \equiv$ degree $m = 0$ zonal
$m \equiv$ order $m = l$ sectoral
$0 < m < l$ tesseral

l even $\Rightarrow$ symmetric

l odd $\Rightarrow$ antisymmetric

$l-m$ zero crossings in colatitude
$l-m+1$ patches in colatitude

$2m$ zero crossings in longitude

$2m$ patches in longitude

wavelength $\lambda = \dfrac{2\pi a}{\sqrt{l(l+1)}} \sim \dfrac{2\pi a}{l+1/2}$

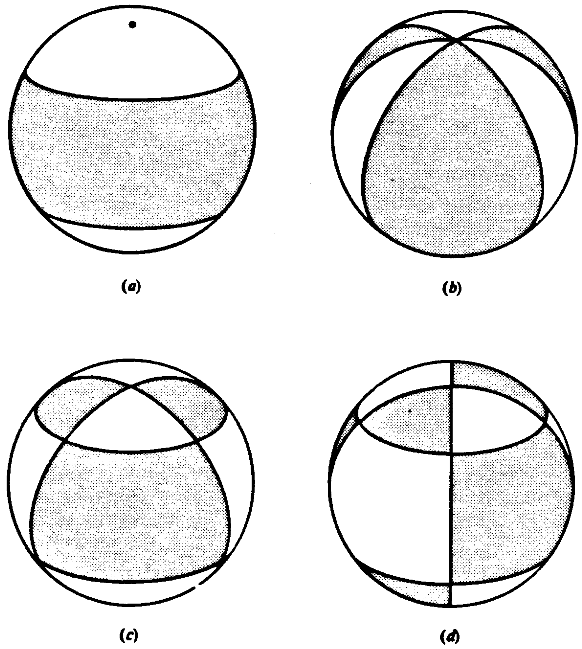


Fig. 2.2. Spherical harmonics. Hatched and blank regions show different signs.

(a) $P_2^0 = \frac{1}{4}(1 + 3\cos 2\vartheta)$,
(b) $P_3^3 \cos 3\varphi = 15 \sin^3 \vartheta \cos 3\varphi$,
(c) $P_4^2 \cos 2\varphi = \frac{15}{16}(3 + 4\cos 2\vartheta - 7\cos 4\vartheta)\cos 2\varphi$,
(d) $P_4^2 \sin 2\varphi = \frac{15}{16}(3 + 4\cos 2\vartheta - 7\cos 4\vartheta)\sin 2\varphi$.

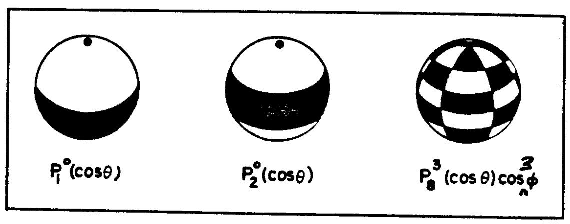


Figure A.2 Examples of surface spherical harmonics. P_1^0 (cos θ) and P_2^0 (cos θ) are dominant terms in the earth's magnetic and gravitational fields respectively. P_8^3 (cos θ cos ϕ) shows a tesseral harmonic.

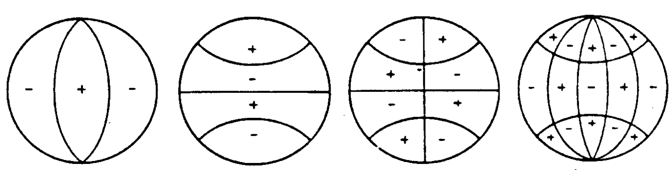


Figure C.1. Examples of spherical harmonics. $m = 0$ gives zonal harmonics, $m = l$ gives sectoral harmonics and the general case $0 < m < l$ are known as tesseral harmonics. Reproduced, by permission, from Kaula (1968).

Expression of a scalar field in spherical harmonics

$$f(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} P_l^m(\cos\theta)\left[a_l^m \cos m\varphi + b_l^m \sin m\varphi\right]$$

Examples: Topography
Gravity
Magnetic field

Spectrum: root mean square coefficient

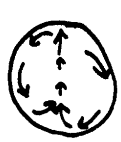
$$S_l^{rms} = \left(\frac{\sum_{m=0}^{l} \left({a_l^m}^2 + {b_l^m}^2 \right)}{(2l+1)} \right)^{1/2}$$

Vector spherical harmonics - eg. displacement
2 types
Poloidal (spheroidal)

$$u_r = U_r(r)\, P_l^m(\cos\theta) \begin{bmatrix} \cos m\varphi \\ \sin m\varphi \end{bmatrix}$$

$$u_\theta = V(r)\, \frac{\partial P_l^m}{\partial\theta} \begin{bmatrix} \cos m\varphi \\ \sin m\varphi \end{bmatrix}$$

$$u_\varphi = \frac{V(r)}{\sin\theta}\, P_l^m \frac{\partial}{\partial\varphi} \begin{bmatrix} \cos m\varphi \\ \sin m\varphi \end{bmatrix}$$


(section)

Toroidal

$$u_\theta = \frac{W(r)}{\sin\theta} P_\ell^m \frac{\partial}{\partial\varphi}\begin{Bmatrix}\cos m\varphi \\ \sin m\varphi\end{Bmatrix}$$

$$u_\varphi = -W(r) \frac{\partial P_\ell^m}{\partial\theta}\begin{bmatrix}\cos m\varphi \\ \sin m\varphi\end{bmatrix}$$

Spectra

KAULA:

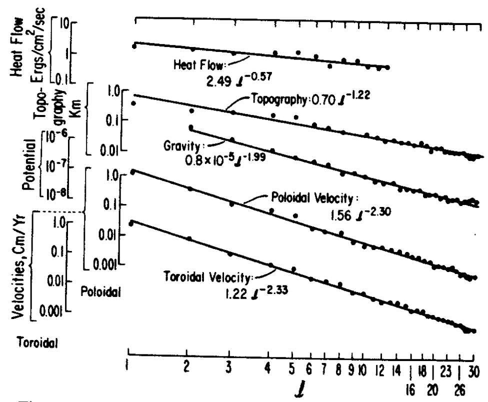


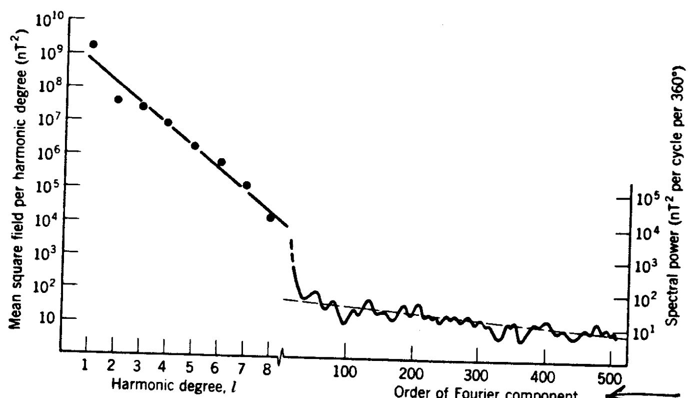
Fig. 1. Spectral magnitudes of observed surface fields

Also: plate converge(nce) spectrum ~ $\ell^{-0.45}$

correlates with gravity field $\ell = 4-9$

Stacy

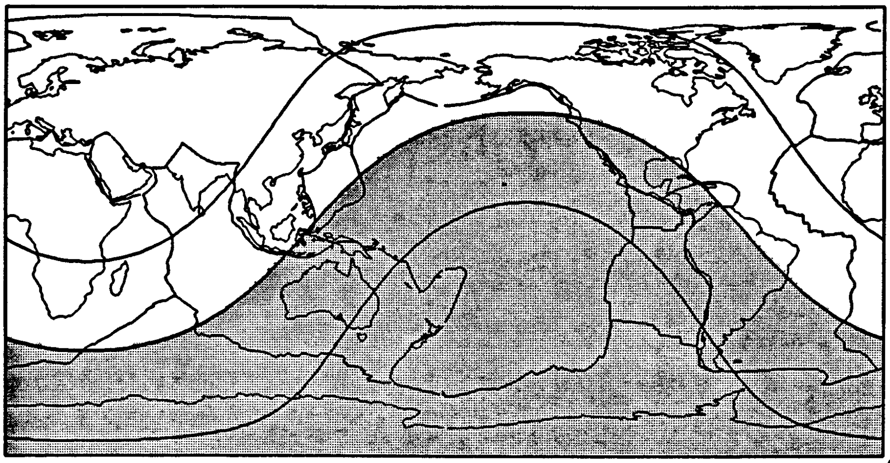
8.1 THE MAIN FIELD 217



asymptotic limit

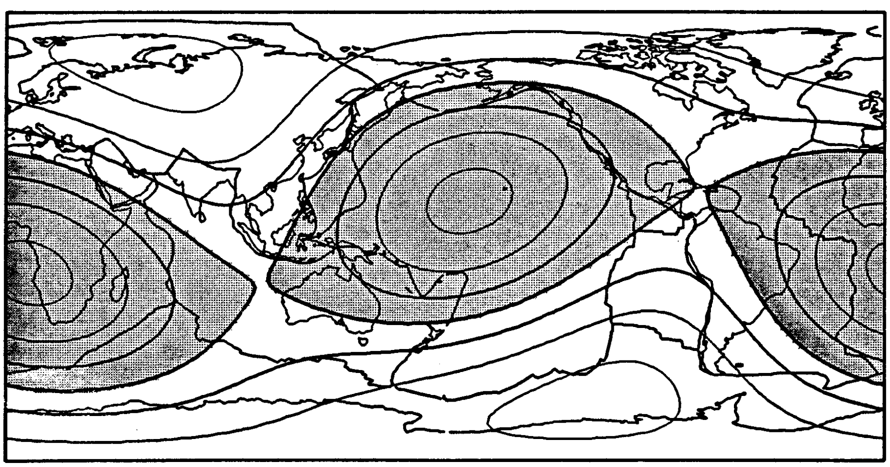
Figure 8.2. Spatial spectrum of the main geomagnetic field. The left part of the figure gives the mean square amplitudes of the field represented by each harmonic degree, for low degrees, according to Lowes (1974) and the right-hand part gives the higher-order terms of a Fourier analysis by Bullard (1967), corrected by Lowes (1974), of a world-encircling line profile by Alldredge et al. (1963).

Global Topography: degree 1-1



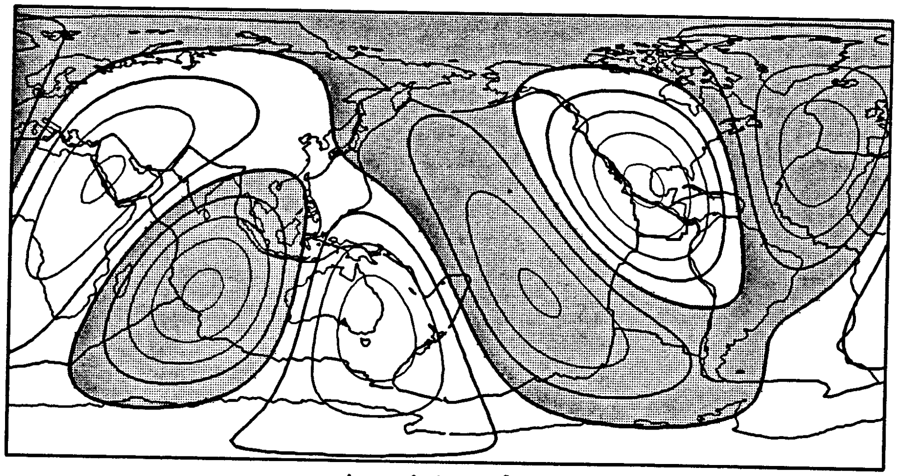
contour interval: 1000 m ; lows shaded

Global Topography: degree 2-2



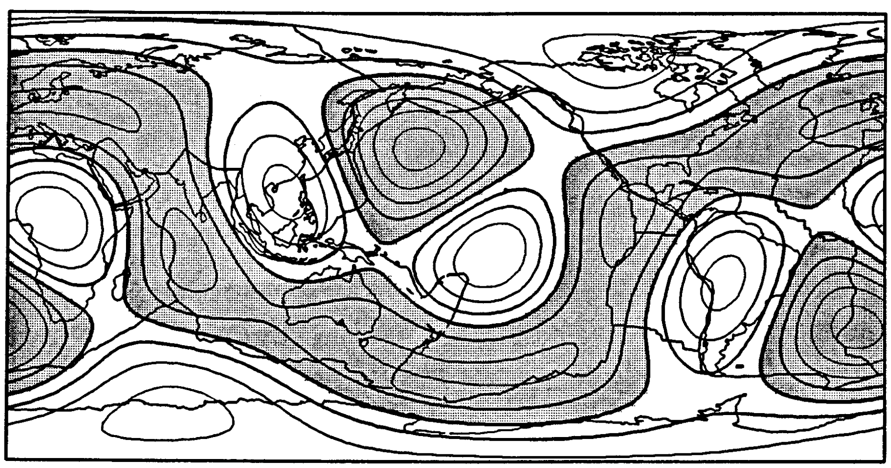
contour interval: 500 m

Global Topography: degree 3-3



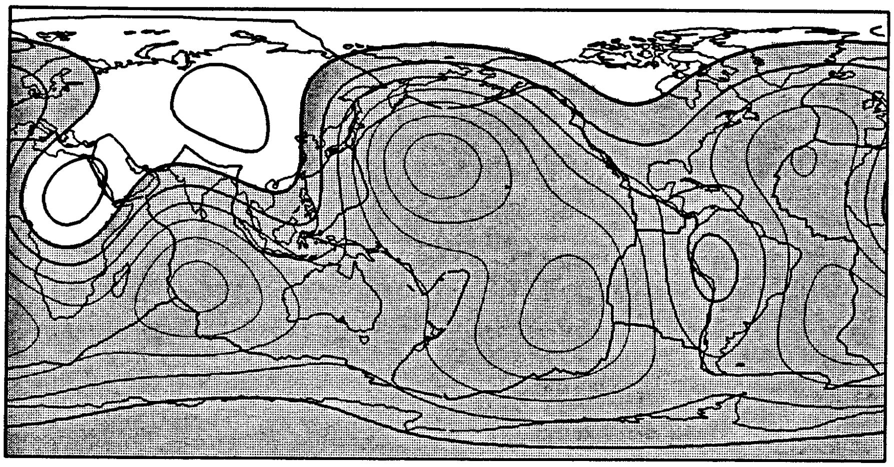
contour interval: 500 m

Global Topography: degree 4-4



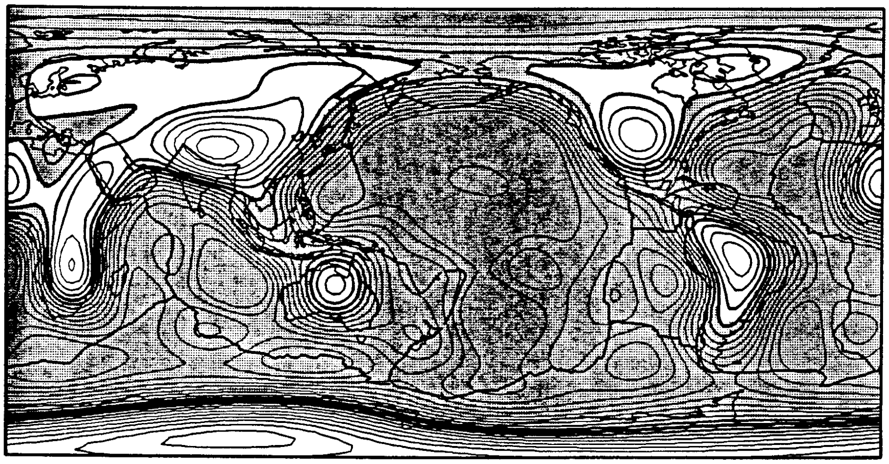
contour interval: 500 m

Global Topography: degree 0-4



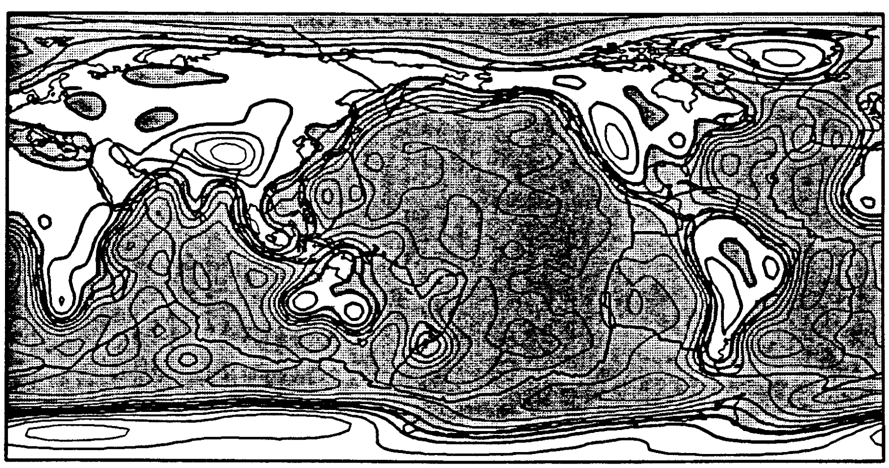
contour interval: 1000 m

Global Topography: degree 0-10



contour interval: 500 m

Global Topography: degree 0-20



contour interval: 1000 m

Relationship between V + g

In spherical harmonics, for $r > a$

$$V(r,\theta,\varphi) = \frac{GM}{r}\left[1 + \sum_{l=2}^{\infty}\left(\frac{a}{r}\right)^l \sum_{m=0}^{l} P_l^m(\cos\theta)\left(C_l^m \cos m\varphi + S_l^m \sin m\varphi\right)\right]$$

$V_0(r)$ $\quad P_0^0 \quad$ why no $l=1$?

$$\underset{\sim}{g} = \nabla V$$

$$g_r = -\frac{GM}{r^2}\left[1 + \sum_{l=2}^{\infty}(l+1)\left(\frac{a}{r}\right)^l \sum_{m=0}^{l} P_l^m(\cos\theta)\left(C_l^m \cos m\varphi + S_l^m \sin m\varphi\right)\right]$$

$$g_\theta = \frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{GM}{r^2}\sum_{l=2}^{\infty}\left(\frac{a}{r}\right)^l \sum_{m=0}^{l} \frac{\partial P}{\partial\theta}\left(C_l^m \cos m\varphi + S_l^m \sin m\varphi\right)\Big]$$

$$g_\varphi = \frac{1}{r\sin\theta}\frac{\partial V}{\partial\varphi} \longrightarrow m\left(-C_l^m \sin m\varphi + S_l^m \cos m\varphi\right)$$

For ∞ series, g + V provide same information!

Usually look at g_r

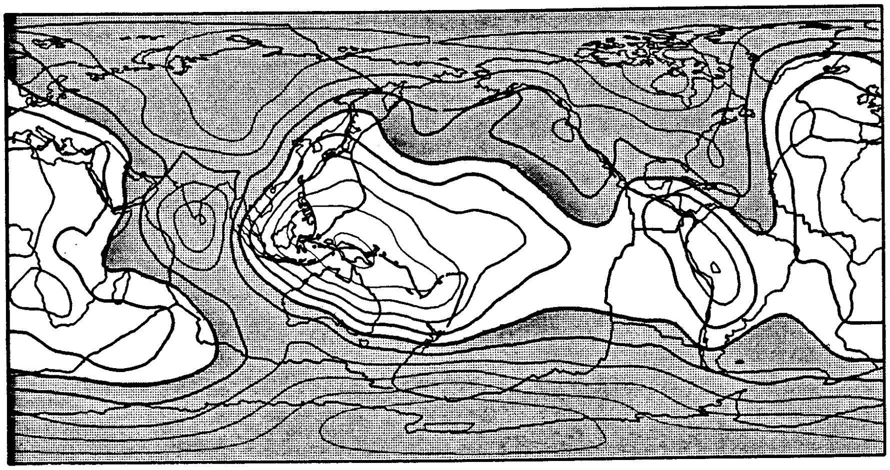
$$g_l^m \sim (l+1)\, V_l^m \implies g \text{ emphasizes}$$

high frequency

V emphasizes low frequency

For ~~perfect~~ coverage, both contain same info.

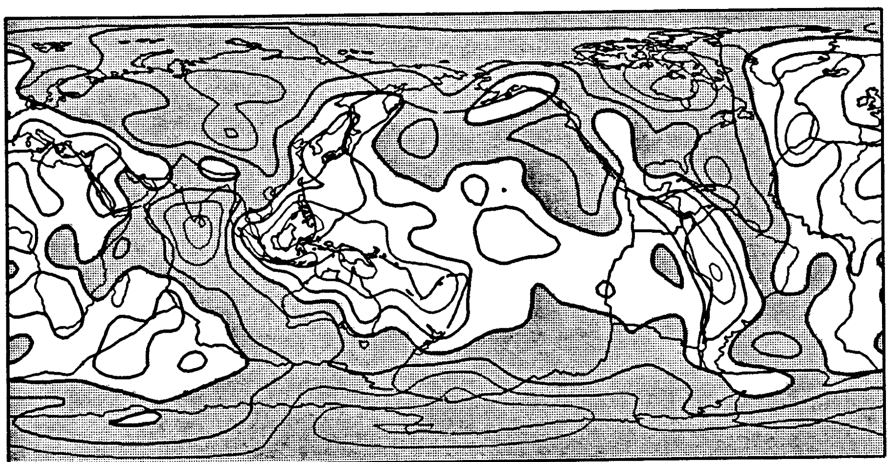
Observed Geoid: degree 2-20



$S_\ell \sim \ell^{-2}$

contour interval: 20 m

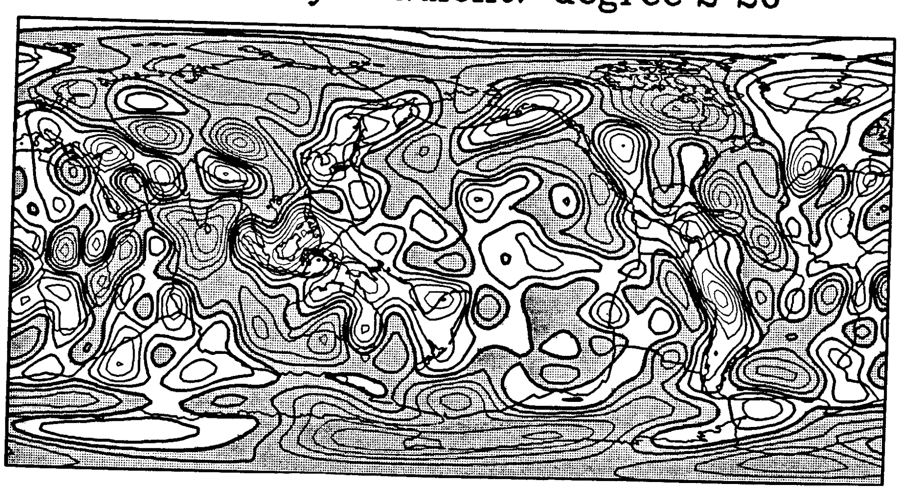
Free Air Gravity: degree 2-20



$S_\ell \sim \ell^{-1}$

contour interval: 20 mgal

Gravity Gradient: degree 2-20



$S \sim \ell^{0}$